

## C-SUPPLEMENTED SUBALGEBRAS OF LIE ALGEBRAS

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**Abstract**

A subalgebra  $B$  of a Lie algebra  $L$  is *c-supplemented* in  $L$  if there is a subalgebra  $C$  of  $L$  with  $L = B + C$  and  $B \cap C \leq B_L$ , where  $B_L$  is the core of  $B$  in  $L$ . This is analogous to the corresponding concept of a c-supplemented subgroup in a finite group. We say that  $L$  is *c-supplemented* if every subalgebra of  $L$  is c-supplemented in  $L$ . We give here a complete characterisation of c-supplemented Lie algebras over a general field.

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**1 Introduction**

The concept of a c-supplemented subgroup of a finite group was introduced by Ballester-Bolínches, Wang and Xiuyun in [2] and has since been studied

by a number of authors. The purpose of this paper is study the corresponding idea for Lie algebras. As we shall see, stronger results can be obtained in this context.

Throughout  $L$  will denote a finite-dimensional Lie algebra over a field  $F$ . If  $B$  is a subalgebra of  $L$  we define  $B_L$ , the *core* (with respect to  $L$ ) of  $B$  to be the largest ideal of  $L$  contained in  $B$ . We say that  $B$  is *core-free* in  $L$  if  $B_L = 0$ . A subalgebra  $B$  of  $L$  is *c-supplemented* in  $L$  if there is a subalgebra  $C$  of  $L$  with  $L = B + C$  and  $B \cap C \leq B_L$ . We say that  $L$  is *c-supplemented* if every subalgebra of  $L$  is c-supplemented in  $L$ . We shall give a complete characterisation of c-supplemented Lie algebras over a general field.

Following [4] we will say that  $L$  is *completely factorisable* if for every subalgebra  $B$  of  $L$  there is a subalgebra  $C$  such that  $L = B + C$  and  $B \cap C = 0$ . It turns out that c-supplemented Lie algebras are intimately related to the completely factorisable ones, and our results generalise some of those obtained in [4]. Incidentally, it is claimed in [4] that if  $F$  has characteristic zero then  $L$  is completely factorisable if and only if the Frattini subalgebra of every subalgebra of  $L$  is trivial. We shall see that this is false.

If  $A$  and  $B$  are subalgebras of  $L$  for which  $L = A + B$  and  $A \cap B = 0$  we will write  $L = A \dot{+} B$ ; if, furthermore,  $A, B$  are ideals of  $L$  we write  $L = A \oplus B$ . The notation  $A \leq B$  will indicate that  $A$  is a subalgebra of  $B$ , and  $A < B$  will mean that  $A$  is a proper subalgebra of  $B$ .

## 2 Preliminary results

First we give some basic properties of c-supplemented subalgebras

**Lemma 2.1** (i) *If  $B$  is c-supplemented in  $L$  and  $B \leq K \leq L$  then  $B$  is c-supplemented in  $K$ .*

(ii) *If  $I$  is an ideal of  $L$  and  $I \leq B$  then  $B$  is c-supplemented in  $L$  if and only if  $B/I$  is c-supplemented in  $L/I$ .*

(iii) *If  $\mathcal{X}$  is the class of all c-supplemented Lie algebras then  $\mathcal{X}$  is subalgebra and factor algebra closed.*

*Proof.*

- (i) Suppose that  $B$  is  $c$ -supplemented in  $L$  and  $B \leq K \leq L$ . Then there is a subalgebra  $C$  of  $L$  with  $L = B + C$  and  $B \cap C \leq B_L$ . It follows that  $K = (B + C) \cap K = B + C \cap K$  and  $B \cap C \cap K \leq B_L \cap K \leq B_K$ , and so  $B$  is  $c$ -supplemented in  $K$ .
- (ii) Suppose first that  $B/I$  is  $c$ -supplemented in  $L/I$ . Then there is a subalgebra  $C/I$  of  $L/I$  such that  $L/I = B/I + C/I$  and  $(B/I) \cap (C/I) \leq (B/I)_{L/I} = B_L/I$ . It follows that  $L = B + C$  and  $B \cap C \leq B_L$ , whence  $B$  is  $c$ -supplemented in  $L$ .  
 Suppose conversely that  $I$  is an ideal of  $L$  with  $I \leq B$  such that  $B$  is  $c$ -supplemented in  $L$ . Then there is a subalgebra  $C$  of  $L$  such that  $L = B + C$  and  $B \cap C \leq B_L$ . Now  $L/I = B/I + (C + I)/I$  and  $(B/I) \cap (C + I)/I = (B \cap (C + I))/I = (I + B \cap C)/I \leq B_L/I = (B/I)_{L/I}$ , and so  $B/I$  is  $c$ -supplemented in  $L/I$ .
- (iii) This follows immediately from (i) and (ii).

The *Frattini ideal* of  $L$ ,  $\phi(L)$ , is the largest ideal of  $L$  contained in all maximal subalgebras of  $L$ . We say that  $L$  is  $\phi$ -free if  $\phi(L) = 0$ . The next result shows that subalgebras of the Frattini ideal of a  $c$ -supplemented Lie algebra  $L$  are necessarily ideals of  $L$ .

**Proposition 2.2** *Let  $B, D$  be subalgebras of  $L$  with  $B \leq \phi(D)$ . If  $B$  is  $c$ -supplemented in  $L$  then  $B$  is an ideal of  $L$  and  $B \leq \phi(L)$ .*

*Proof.* Suppose that  $L = B + C$  and  $B \cap C \leq B_L$ . Then  $D = D \cap L = D \cap (B + C) = B + D \cap C = D \cap C$  since  $B \leq \phi(D)$ . Hence  $B \leq D \leq C$ , giving  $B = B \cap C \leq B_L$  and  $B$  is an ideal of  $L$ . It then follows from [6, Lemma 4.1] that  $B \leq \phi(L)$ .

The Lie algebra  $L$  is called *elementary* if  $\phi(B) = 0$  for every subalgebra  $B$  of  $L$ ; it is an *E-algebra* if  $\phi(B) \leq \phi(L)$  for all subalgebras  $B$  of  $L$ . Then we have the following useful corollary.

**Corollary 2.3** *If  $L$  is  $c$ -supplemented then  $L$  is an E-algebra.*

*Proof.* Simply put  $B = \phi(D)$  in Proposition 2.2.

It is clear that if  $L$  is completely factorisable then it is  $c$ -supplemented. However, the converse is false. Every completely factorisable Lie algebra must be  $\phi$ -free, whereas the same is not true for  $c$ -supplemented algebras. For example, the three-dimensional Heisenberg algebra is  $c$ -supplemented, as will be clear from the next result which gives the true relationship between these two classes of algebras.

**Proposition 2.4** *Let  $L$  be a Lie algebra. Then the following are equivalent:*

- (i)  $L$  is  $c$ -supplemented.
- (ii)  $L/\phi(L)$  is completely factorisable and every subalgebra of  $\phi(L)$  is an ideal of  $L$ .

*Proof.* (i)  $\Rightarrow$  (ii): Suppose first that  $L$  is  $\phi$ -free and  $c$ -supplemented, and let  $B$  be a subalgebra of  $L$ . Then there is a subalgebra  $C$  of  $L$  such that  $L = B + C$ . Choose  $D$  to be a subalgebra of  $L$  minimal with respect to  $L = B + D$ . Then  $B \cap D \leq \phi(D)$ , by [6, Lemma 7.1], whence  $B \cap D = 0$  since  $L$  is elementary, by Corollary 2.3. Hence  $L$  is completely factorisable, and (ii) follows from Lemma 2.1(iii) and Proposition 2.2.

(ii)  $\Rightarrow$  (i): Suppose that (ii) holds and let  $B$  be a subalgebra of  $L$ . Then there is a subalgebra  $C/\phi(L)$  of  $L/\phi(L)$  such that  $L/\phi(L) = ((B + \phi(L))/\phi(L)) + (C/\phi(L))$  and  $0 = ((B + \phi(L))/\phi(L)) \cap (C/\phi(L)) = (B \cap C + \phi(L))/\phi(L)$ . Hence  $L = B + C$  and  $B \cap C \leq \phi(L)$ , so  $B \cap C$  is an ideal of  $L$  and  $B \cap C \leq B_L$ ; that is,  $L$  is  $c$ -supplemented.

Note that if  $L$  is the three-dimensional Heisenberg algebra, then condition (ii) in the above result holds, since  $\phi(L) = L^2$  is one dimensional and  $L/\phi(L)$  is abelian. Finally we shall need the following result concerning direct sums of completely factorisable Lie algebras.

**Lemma 2.5** *If  $A$  and  $B$  are completely factorisable, then so is  $L = A \oplus B$ .*

*Proof.* Suppose that  $A, B$  are completely factorisable and put  $L = A \oplus B$ . Let  $U$  be a subalgebra of  $L$ . If  $A \leq U$ , then  $U = A \oplus (B \cap U)$ . Since  $B$  is completely factorisable there is a subalgebra  $C$  of  $B$  such that  $B = B \cap U + C$  and  $U \cap C = B \cap U \cap C = 0$ . Hence  $L = U \dot{+} C$ .

Now  $A \leq A + U$  so, by the above, there is a subalgebra  $C$  of  $B$  with  $L = A + U + C$  and  $(A + U) \cap C = 0$ . Moreover, since  $A$  is completely

factorisable, there is a subalgebra  $D$  of  $A$  such that  $A = A \cap U + D$  and  $U \cap D = A \cap U \cap D = 0$ . It follows that  $L = U + (D \oplus C)$  and  $U \cap (D + C) \leq U \cap [(A + U) \cap (D + C)] = U \cap [D + (A + U) \cap C] = U \cap D = 0$ . It follows that  $L$  is completely factorisable.

Note that the corresponding result for  $c$ -supplemented Lie algebras is false. For, let  $L_1 = Fx + Fy + Fz$  with  $[x, y] = -[y, x] = y + z$ ,  $[x, z] = -[z, x] = z$  and all others products equal to zero. Then it is straightforward to check that  $\phi(L_1) = Fz$  and that  $L_1$  is  $c$ -supplemented. Now take  $L$  to be a direct sum of two copies of  $L_1$ : say,  $L = A \oplus B$  where  $A = Fx + Fy + Fz$ ,  $B = Fa + Fb + Fc$ ,  $[x, y] = -[y, x] = y + z$ ,  $[x, z] = -[z, x] = z$ ,  $[a, b] = -[b, a] = b + c$ ,  $[a, c] = -[c, a] = c$  and all others products equal to zero. Suppose that  $F(z + c)$  is  $c$ -supplemented in  $L$ . Then there is a subalgebra  $M$  of  $L$  with  $L = F(z + c) + M$  and  $F(z + c) \cap M \leq (F(z + c))_L$ . If  $z + c \notin M$  then  $M$  is a maximal subalgebra of  $L$ , contradicting the fact that  $z + c \in (\phi(A) \oplus \phi(B)) = \phi(L)$ , by [6, Theorem 4.8]. It follows that  $z + c \in M$ , whence  $F(z + c)$  is an ideal of  $L$ . But  $[x, z + c] = z \notin F(z + c)$ , a contradiction. Thus  $L$  is not  $c$ -supplemented in  $L$ .

### 3 The structure theorems

We can now give the main structure theorems for  $c$ -supplemented Lie algebras. First we determine the solvable ones.

**Theorem 3.1** *Let  $L$  be a solvable Lie algebra. Then the following are equivalent:*

- (i)  $L$  is  $c$ -supplemented.
- (ii)  $L$  is supersolvable and every subalgebra of  $\phi(L)$  is an ideal of  $L$ .

*Proof.* (i)  $\Rightarrow$  (ii): We have that every subalgebra of  $\phi(L)$  is an ideal of  $L$  by Proposition 2.4, so we have only to show that  $L$  is supersolvable. Let  $L$  be a minimal counter-example. Then all proper subalgebras and factor algebras of  $L$  are supersolvable, by Lemma 2.1(iii). If we can show that all maximal subalgebras have codimension one in  $L$ , we shall have the desired contradiction, by [3, Theorem 7]; so let  $M$  be any maximal subalgebra of  $L$ . Since the result is clear if  $M_L \neq 0$ , we may assume that  $M_L = 0$ .

Pick a minimal ideal  $A$  of  $L$ . Then  $L = A \dot{+} M$  and  $A$  is the unique minimal ideal of  $L$ , by [7, Lemma 1.4]. Let  $a \in A$ . Then  $Fa$  is  $c$ -supplemented in  $L$ , and so there is a subalgebra  $B$  of  $L$  such that  $L = Fa + B$  and  $Fa \cap B \leq (Fa)_L$ . If  $a \in B$  then  $Fa$  is an ideal of  $L$ , whence  $A = Fa$  and  $M$  has codimension one in  $L$ .

So suppose that  $L = Fa \dot{+} B$ . Since  $A \not\leq B$  we have  $B_L = 0$ . But then  $L = A \dot{+} B$  by [7, Lemma 1.4] again. It follows that  $\dim A = 1$  and  $M$  has codimension one in  $L$ .

(ii)  $\Rightarrow$  (i): By Proposition 2.4, it suffices to show that if  $L$  is supersolvable and  $\phi$ -free then it is completely factorisable. Let  $L$  be a minimal counterexample. Then  $L$  is elementary, by [5, Theorem 1], and so every proper subalgebra of  $L$  is completely factorisable. Also  $L = A \dot{+} B$  where  $A = Fa_1 \oplus \dots \oplus Fa_n$  is the abelian socle of  $L$  and  $B$  is abelian, by [7, Theorem 7.3]. Let  $U$  be a subalgebra of  $L$ . If  $A \leq U$  it is clear that there is a subalgebra  $C$  of  $L$  such that  $L = U + C$  and  $U \cap C = 0$ . So suppose that  $a_i \notin U$  for some  $1 \leq i \leq n$ ; we may as well assume that  $i = 1$ . Then  $L/Fa_1 \cong (Fa_2 \oplus \dots \oplus Fa_n) \dot{+} B$ , which is a proper subalgebra of  $L$  and so is completely factorisable. Hence there is a subalgebra  $C$  of  $L$  such that  $L/Fa_1 = ((U + Fa_1)/Fa_1) + (C/Fa_1)$  and  $Fa_1 = (U + Fa_1) \cap C = U \cap C + Fa_1$ . It follows that  $L = U + C$  and  $U \cap C \leq Fa_1$ . But  $a_1 \notin U \cap C$  so  $U \cap C = 0$  and  $L$  is completely factorisable, a contradiction.

We shall need the following classification of Lie algebras with core-free subalgebras of codimension one which is given by Amayo in [1].

**Theorem 3.2** ([1, Theorem 3.1]) *Let  $L$  have a core-free subalgebra of codimension one. Then either (i)  $\dim L \leq 2$ , or else (ii)  $L \cong L_m(\Gamma)$  for some  $m$  and  $\Gamma$  satisfying certain conditions (see [1] for details).*

We shall also need the following properties of  $L_m(\Gamma)$  which are given by Amayo in [1].

**Theorem 3.3** ([1, Theorem 3.2])

- (i) *If  $m > 1$  and  $m$  is odd, then  $L_m(\Gamma)$  is simple and has only one subalgebra of codimension one.*
- (ii) *If  $m > 1$  and  $m$  is even, then  $L_m(\Gamma)$  has a unique proper ideal of codimension one, which is simple, and precisely one other subalgebra of codimension one.*

- (iii)  $L_1(\Gamma)$  has a basis  $\{u_{-1}, u_0, u_1\}$  with multiplication  $[u_{-1}, u_0] = u_{-1} + \gamma_0 u_1$  ( $\gamma_0 \in F, \gamma_0 = 0$  if  $\Gamma = \{0\}$ ),  $[u_{-1}, u_1] = u_0, [u_0, u_1] = u_1$ .
- (iv) If  $F$  has characteristic different from two then  $L_1(\Gamma) \cong L_1(0) \cong sl_2(F)$ .
- (v) If  $F$  has characteristic two then  $L_1(\Gamma) \cong L_1(0)$  if and only if  $\gamma_0$  is a square in  $F$ .

The above properties enable us to determine which of the algebras  $L_m(\Gamma)$  are c-supplemented.

**Proposition 3.4** *If  $L \cong L_m(\Gamma)$  then  $L$  is c-supplemented if and only if  $L \cong L_1(0)$  and  $F$  has characteristic different from two.*

*Proof.* Suppose that  $L \cong L_m(\Gamma)$  and  $L$  is c-supplemented, and let  $x \in L$ . Then there is a subalgebra  $M_1$  of  $L$  such that  $L = Fx + M_1$ , and  $Fx \cap M_1 \leq (Fx)_L = 0$ , since  $L_m(\Gamma)$  has no one-dimensional ideals. Choose  $y \in M_1$ . Then, similarly, there is a subalgebra  $M_2$  of codimension one in  $L$  such that  $L = Fy + M_2$  and  $M_1 \neq M_2$ . Since  $L = M_1 + M_2$  we have that  $M_1 \cap M_2 \neq 0$ . Let  $z \in M_1 \cap M_2$ . Then there is a subalgebra  $M_3$  of codimension one in  $L$  such that  $L = Fz + M_3$ , so  $L$  has at least three subalgebras of codimension one in  $L$ . It follows from Theorem 3.3 that  $m = 1$ .

Suppose that  $L \not\cong L_1(0)$ . Then  $F$  has characteristic two and  $\gamma_0$  is not a square in  $F$ . Since  $L$  is completely factorisable there is a two-dimensional subalgebra  $M$  of  $L$  such that  $L = Fu_1 + M$ . It follows that  $M = F(u_{-1} + \alpha u_1) + F(u_0 + \beta u_1)$  for some  $\alpha, \beta \in F$ . But then  $[u_{-1} + \alpha u_1, u_0 + \beta u_1] \in M$  shows that  $\gamma_0 = \beta^2$ , a contradiction. A further straightforward calculation shows that if  $L \cong L_1(0)$  and  $F$  has characteristic two, then  $Fu_1$  is contained in every maximal subalgebra of  $L$ , and so has no c-supplement in  $L$ .

Conversely, suppose that  $L \cong L_1(0)$  and  $F$  has characteristic different from two. Then  $L \cong sl_2(F)$ , by Theorem 3.3 (iv) and it is easy to check that  $L$  is c-supplemented.

We can now determine the simple and semisimple c-supplemented Lie algebras.

**Corollary 3.5** *If  $L$  is simple then  $L$  is c-supplemented if and only if  $L \cong L_1(0)$  and  $F$  has characteristic different from two.*

*Proof.* Let  $L$  be simple and  $c$ -supplemented. Then  $L$  has a core-free maximal subalgebra of codimension one in  $L$  and so  $L \cong L_m(\Gamma)$ , by Theorem 3.2. The result now follows from Proposition 3.4.

Notice, in particular, that  $sl_2(F)$  is the only simple completely factorisable Lie algebra over any field. However, this is not the only simple elementary Lie algebra, even over a field of characteristic zero: over the real field every compact simple Lie algebra, and  $so(n, 1)$  for  $n > 3$ , for example, are elementary, as is shown in [8, Theorem 5.1]. This justifies the assertion made at the end of the third paragraph of the introduction.

**Proposition 3.6** *Let  $L$  be a semisimple Lie algebra over a field  $F$ . Then the following are equivalent:*

- (i)  $L$  is  $c$ -supplemented.
- (ii)  $L = S_1 \oplus \dots \oplus S_n$  where  $S_i \cong sl_2(F)$  for  $1 \leq i \leq n$  and  $F$  has characteristic different from two.

*Proof.* (i)  $\Rightarrow$  (ii): Let  $L$  be semisimple and  $c$ -supplemented and suppose the result holds for all such algebras of dimension less than  $\dim L$ . Then  $\phi(L) = 0$ , since  $\phi(L)$  is nilpotent, and so  $L$  is completely factorisable. Let  $A$  be a minimal ideal of  $L$  and pick  $a \in A$ . Let  $M$  be a subalgebra of  $L$  such that  $L = Fa + M$  and put  $B = A + M_L$ . Then  $M_L < B$  and  $A \cap M_L = 0$ , since  $a \notin M_L$ . If  $\dim L/M_L \leq 2$  then  $A$  is abelian, contradicting the fact that  $L$  is semisimple. It follows from Theorem 3.2 and Proposition 3.4 that  $L/M_L \cong L_1(0)$ , whence  $B = L$  and  $L = A \oplus M_L$ . Since  $A, M_L$  are semisimple and  $c$ -supplemented the result follows.

(ii)  $\Rightarrow$  (i): The converse follows from Corollary 3.5 and Lemma 2.5.

Finally we have the main classification theorem.

**Theorem 3.7** *Let  $L$  be Lie algebra. Then the following are equivalent:*

- (i)  $L$  is  $c$ -supplemented.
- (ii)  $L/\phi(L) = R \oplus S$  where  $R$  is supersolvable and  $\phi$ -free,  $S$  is given by Proposition 3.6, and every subalgebra of  $\phi(L)$  is an ideal of  $L$ .



*Proof.* (i)  $\Rightarrow$  (ii): Factor out  $\phi(L)$  so that  $L$  is  $\phi$ -free and  $c$ -supplemented and hence completely factorisable, by Proposition 2.4. Then  $L = R \dot{+} S$  where  $R$  is the radical of  $L$  and  $S$  is semisimple. It suffices to show that  $SR = 0$ ; the rest follows from Lemma 2.1, Corollary 2.3, Proposition 2.4, Theorem 3.1 and Proposition 3.6. Suppose there is  $0 \neq x \in L^{(3)} \cap R$ . Then there is a subalgebra  $M$  of  $L$  such that  $L = Fx \dot{+} M$  and  $L/M_L$  is given by Theorem 3.2. If  $L/M_L \cong L_m(\Gamma)$  then  $L/M_L$  is simple, by Proposition 3.4, and  $M_L < R + M_L$ , so  $L = R + M_L$ . But then  $L/M_L$  is solvable, a contradiction. It follows that  $\dim L/M_L \leq 2$ , whence  $x \in L^{(3)} \cap R \leq L^{(3)} \leq M_L \leq M$ , a contradiction. Hence  $L^{(3)} \cap R = 0$ . But  $SR = S^2R \leq S(SR) = S^2(SR) \leq L^{(3)} \cap R = 0$ , as required.

(ii)  $\Rightarrow$  (i): This follows from Proposition 2.4, Lemma 2.5, Theorem 3.1 and Proposition 3.6.

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